# Homework 3

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## 1 Strassen's algorithm

(a)

$$\begin{bmatrix}1 & 3\\7 & 5\end{bmatrix} \odot \begin{bmatrix}6 & 8\\4 & 2\end{bmatrix}$$

$S_1 = 6, S_2 = 4$
$S_3 = 12, S_4 = -2$
$S_5 = 5, S_6 = 8$
$S_9 = -6, S_{10} = 14$

$P_1 = 1 * 6 = 6, P_2 = 4 * 2 = 8$
$P_3 = 6 * 12 = 72, P_4 = -2 * 5 = -10$
$P_5 = 6 * 8 = 48, P_6 = -2 * 6 = -12$
$P_7 = -6 * 14 = -84$

$$= \begin{bmatrix}c_{11} & c_{12}\\c_{21} & c_{22}\end{bmatrix} = \begin{bmatrix}(P_5 + P_4 - P_2 + P_6) & (P_1 + P_2)\\(P_3 + P_4) & (P - 5 + P_1 - P_3 + P_7)\end{bmatrix}$$

$$= \begin{bmatrix}(48 + (-10) - 8 + (-12)) & (6 + 8)\\(72 + (-10)) & (48 + 6 - 72 - (-84))\end{bmatrix} = \begin{bmatrix}18 & 14\\62 & 66\end{bmatrix}$$

(b)

$$A \odot B = \begin{cases} A * B & \text{if } A_{rows} = 1 \\ \begin{bmatrix}c_{11} & c_{12}\\c_{21} & c_{22}\end{bmatrix} & \text{otherwise}\end{cases}$$

Where: $A = \begin{bmatrix}A_{11} & A_{12}\\A_{21} & A_{22}\end{bmatrix}$

$B = \begin{bmatrix}B_{11} & B_{12}\\B_{21} & B_{22}\end{bmatrix}$

$c_{11} = A_{11} \odot B_{11} + A_{12} \odot B_{21}$
$c_{12} = A_{11} \odot B_{12} + A_{12} \odot B_{22}$
$c_{21} = A_{21} \odot B_{11} + A_{22} \odot B_{21}$
$c_{22} = A_{21} \odot B_{12} + A_{22} \odot B_{22}$

**Algorithm 1** Strassen's Algorithm

**procedure** STRASSEN(A,B):
  $n \leftarrow$ number of rows in $A$
  $C \leftarrow$ *new n by n matrix*
  **if** $n = 1$ **then**
    $c \leftarrow A[1][1] * B[1][1]$
  **else**
    Sub partition A into 4 equal matrix quadrants A11, A12, A21, A22
    Sub partition B into 4 equal matrix quadrants B11, B12, B21, B22
    $s1 \leftarrow B12 - B22$
    $s2 \leftarrow A11 + A12$
    $s3 \leftarrow A21 + A22$
    $s4 \leftarrow B21 - B11$
    $s5 \leftarrow A11 + A22$
    $s6 \leftarrow B11 + B22$
    $s7 \leftarrow A12 - A22$
    $s8 \leftarrow B21 + B22$
    $s9 \leftarrow A11 - A21$
    $s10 \leftarrow B11 + B12$
    $p1 \leftarrow Strassen(A11, S1)$
    $p2 \leftarrow Strassen(S2, B22)$
    $p3 \leftarrow Strassen(S3, B11)$
    $p4 \leftarrow Strassen(A22, S4)$
    $p5 \leftarrow Strassen(S5, S6)$
    $p6 \leftarrow Strassen(S7, S8)$
    $p7 \leftarrow Strassen(S9, S10)$
    $C[1][1] \leftarrow p5 + p4 - p2 + p6$
    $C[1][2] \leftarrow p1 + p2$
    $C[2][1] \leftarrow p3 + p4$
    $C[2][2] \leftarrow p5 + p1 - p3 + p7$
  **return** C

(c) $T(1) = 1$
$T(n) = 7T(\frac{n}{2}) + \frac{9}{2}n^2$
$T(2^0) = 1$
$T(2^m) = 7T(2^{m-1}) + \frac{9}{2}(2^m)^2$
$= 7(7T(n^{m-2}) + \frac{9}{2}(2^{m-1})^2) + \frac{9}{2}(2^m)^2$
$= 7^2T(n^{m-2}) + 7 * \frac{9}{2}(2^{m-1})^2 + 7^0 * \frac{9}{2}(2^m)^2$
$= 7^2(7T(2^{m-3}) + \frac{9}{2}(2^{m-2})^2) + 7 * \frac{9}{2}(2^{m-1})^2 + 7^0 * \frac{9}{2}(2^m)^2$
$= 7^3T(2^{m-3}) + 7^2\frac{9}{2}(2^{m-2})^2 + 7 * \frac{9}{2}(2^{m-1})^2 + 7^0 * \frac{9}{2}(2^m)^2$
$= 7^kT(2^{m-k}) + 7^{(k-1)}\frac{9}{2}(2^{m-(k-1)})^2 + ... + 7^{k-k} * \frac{9}{2}(2^{m-(k-k)})^2$

Let $m = k$
$T(2^m) = 7^mT(2^{m-m}) + 7^{(m-1)}\frac{9}{2}(2^{m-(m-1)})^2 + ... + 7^{m-m} * \frac{9}{2}(2^{m-(m-m)})^2$
$T(2^m) = 7^m + 7^{(m-1)}\frac{9}{2}(2^1)^2 + ... + 7^0 * \frac{9}{2}(2^m)^2$
$= 7^m + \frac{9}{2}\sum_{k=0}^{m-1} 7^k(2^{m-k})^2$
$= \frac{9}{2}\sum_{k=0}^{m} 7^k(2^{2(m-k)}) - \frac{9}{2}7^m$

$= \frac{9}{2}\sum_{k=0}^{m} 7^k(2^{2m})(2^{-2k}) - \frac{9}{2}7^m$
$= (2^{2m})\frac{9}{2}\sum_{k=0}^{m} 7^k(2^{-2k}) - \frac{9}{2}7^m + 7^m$
$= (2^{2m})\frac{9}{2}\sum_{k=0}^{m} \frac{7^k}{2^{2k}}) - \frac{7}{2}7^m$
$= (2^{2m})\frac{9}{2}\sum_{k=0}^{m}(\frac{7}{4})^k - \frac{7}{2}7^m$
$= (2^{2m})\frac{9}{2}(\frac{\frac{7}{4}^{m+1}-1)}{1.75-1}) - \frac{7}{2}7^m$
$= (2^m)^2\frac{9}{2}\frac{4}{3}((\frac{7}{4})^{m+1} - 1) - (\frac{7}{2})7^m$
$= 6(2^m)^2((\frac{7}{4})^{m+1} - 1) - (\frac{7}{2})7^m$
$= 6(2^m)^2\frac{7}{4} - 6(2^m)^2 - (\frac{7}{2})7^m$
$= \frac{21*7^m}{2} - 6(2^m)^2 - (\frac{7}{2})7^m$
$= 7 * 7^m - 6 * 4^m$
$T(n) = 7 * n^{lg(7)} - 6 * n^2$
$T(n) \approx 7 * n^{2.81} - 6 * n^2$

(d) Modifying Strassen's Algorithm

To make Strassen's algorithm to work with any n x n matrix we would pad the two matrices being multiplied with zeros to become two m x m matrix where m is a power of 2. The answer would be the result but only taking out the n x n section out of the m x m product.

To prove that this is still asymptotically equal we will demonstrate that at most the dimensions of the matrix being multiply double.

let m = length of new padded matrices being multiplied
let n = length of original matrices being multiplied
Since: $2^{k-1} < n < 2^k = m$
Note: $2n > 2^{k+1} > m$
Hence: $T(n) \in \Theta((2n)^{lg(7)}) = \Theta(n^{lg7})$

(e) $(a + bi)(c + di)$
$= ac + adi + bci + bdi^2$
$= ac + adi + bci - bd$
$RealPart = ac - bd$
$ImaginaryPart = (a + b)(c + d) - ac - bd$

Consider:
$A_1 = ac$
$A_2 = bd$
$A_3 = (a + b)(c + d)$

Then:
Real Part = $A_1 - A_2$
Imaginary Part = $A_3 - A_1 - A_2$

## 2 Recurrence

(a) **Lemma 1**: For any n $\in \mathbb{N}$, $\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$

**Proof By Cases:**
**Even Case:**
n can be represented as 2m for some value of m
$LHS = \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{2m+1}{2} \right\rfloor$
$= \left\lfloor m + \frac{1}{2} \right\rfloor = m$
$RHS = \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2m}{2} \right\rceil = \lceil m \rceil = m = LHS$
**Odd Case:**
n can be represented as (2m + 1) for some value of m
$LHS = \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{2m+1+1}{2} \right\rfloor$
$= \lfloor m+1 \rfloor = m+1$
$RHS = \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2m+1}{2} \right\rceil = \left\lceil m + \frac{1}{2} \right\rceil = m+1 = LHS$

□

(b) **Lemma 2**: For any n $\in \mathbb{N}$, $\left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n+1}{2} \right\rceil$

**Proof By Cases:**
**Even Case:**
n can be represented as 2m for some value of m
$LHS = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lfloor \frac{2m}{2} \right\rfloor + 1$
$= \lfloor m \rfloor = m+1$
$RHS = \left\lceil \frac{n+1}{2} \right\rceil = \left\lceil \frac{2m+1}{2} \right\rceil = \left\lceil m + \frac{1}{2} \right\rceil = m+1 = LHS$
**Odd Case:**
n can be represented as (2m + 1) for some value of m
$LHS = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lfloor \frac{2m+1}{2} \right\rfloor + 1 = \left\lfloor m + \frac{1}{2} \right\rfloor + 1$
$= \lfloor m \rfloor + 1 = m+1$
$RHS = \left\lceil \frac{n+1}{2} \right\rceil = \left\lceil \frac{2m+1+1}{2} \right\rceil = \lceil m+1 \rceil = m+1 = LHS$

□

(c) **Let:** $T(1) = 0, T(n) = T(\left\lfloor \frac{n}{2} \right\rfloor) + T(\left\lceil \frac{n}{2} \right\rceil) + n$
**Let:** $D(n) = T(n+1) - T(n)$

**Lemma 3:** $D(1) = 2$
**Direct Proof:**
$D(1) = T(2) - T(1)$
$= T(\left\lfloor \frac{2}{2} \right\rfloor) + T(\left\lceil \frac{2}{2} \right\rceil) + 2 - T(0)$
$= T(1) + T(1) + 2 - T(0)$
$= 2$
$\square$

**Lemma 4:** $D(n) = D(\left\lfloor \frac{n}{2} \right\rfloor) + 1$
**Direct Proof:**
$D(n) = T(n+1) - T(n)$
$= T(\left\lfloor \frac{n+1}{2} \right\rfloor) + T(\left\lceil \frac{n+1}{2} \right\rceil) + (n+1) - T(n)$
$= T(\left\lceil \frac{n}{2} \right\rceil) + T(\left\lfloor \frac{n}{2} \right\rfloor + 1) + (n+1) - T(n)$ *by lemma 1, 2*
$= T(\left\lceil \frac{n}{2} \right\rceil) + T(\left\lfloor \frac{n}{2} \right\rfloor + 1) + (n+1) - T(\left\lfloor \frac{n}{2} \right\rfloor) - T(\left\lceil \frac{n}{2} \right\rceil) - n$
$= T(\left\lfloor \frac{n}{2} \right\rfloor + 1) - T(\left\lfloor \frac{n}{2} \right\rfloor) + 1$
$= D(\left\lfloor \frac{n}{2} \right\rfloor) + 1$
$\square$

(d) **Lemma 5:** For any $n \in \mathbb{N}$, if $n > 1$ and n is even then :
$\left\lfloor lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor = \lfloor\ lg(n) - 1 \rfloor$
**Direct Proof:**
Suppose that n is even:
$\left\lfloor lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor = \left\lfloor lg(\frac{n}{2}) \right\rfloor$ *since n is even*
$= \lfloor lg(n) - lg(2) \rfloor$
$= \lfloor lg(n) - 1 \rfloor$
$= \lfloor lg(n) \rfloor - 1$
$\square$

**Lemma 6:** For any $m \in \mathbb{N}$, if $m > 0$ then:
$\lfloor lg(\lfloor 2m+1 \rfloor) \rfloor = \lfloor\ lg(2m) \rfloor$
**Direct Proof:**
Let $k = \lfloor lg(\lfloor 2m+1 \rfloor) \rfloor$
$\lfloor lg(\lfloor 2m+1 \rfloor) \rfloor = k \rightarrow k \leq lg(2m+1) < k+1$
$\rightarrow 2^k \leq 2m+1 < 2^{k+1}$
$\rightarrow 2^k - 1 \leq 2m < 2^{k+1} - 1$*by transitivity*
$\rightarrow 2^k - 1 \leq 2m < 2^{k+1}$
$\rightarrow 2^k \leq 2m < 2^{k+1}$*Since 2m is even*
$\rightarrow k \leq lg(2m) < k+1$
$\rightarrow \lfloor\ lg(2m) \rfloor = k$
$\square$

**Lemma 7:** For any $n \in \mathbb{N}$, if $n > 1$ and n is odd then :
$\left\lfloor lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor = \lfloor lg(n) - 1 \rfloor$
**Direct Proof:**
$\left\lfloor lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor = \left\lfloor lg(\frac{n-1}{2}) \right\rfloor$ *Gets an even number since n is odd*
$= \lfloor lg(n-1) - lg(2) \rfloor$
$= \lfloor lg(n-1) - 1 \rfloor$
$= \lfloor lg(n-1) \rfloor - 1$
$= \lfloor lg(n) \rfloor - 1$ *By lemma 6*
□

**Corollary 1:** By lemmas 5 and 7, for any $n \in \mathbb{N}$, if $n > 1$ then :
$\left\lfloor lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor = \lfloor lg(n) - 1 \rfloor$

**Lemma 8:** For any for any $n \in \mathbb{N}$, if $n > 1$ then $D(n) = \lfloor lg(n) \rfloor + 2$
**Proof via Strong Induction:**
**base case: n =1**
$D(1) = 2$ *lemma 3*
$D(1) = \lfloor lg(1) \rfloor + 2$
$= 0 + 2 = 2$
**Inductive Step:** Assume proposition holds up to but not including n.
Show that n follows.
$D(n) = D(\left\lfloor \frac{n}{2} \right\rfloor) + 1$
$= \left\lfloor lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor + 2 + 1$ *By I.H*
$= \lfloor lg(n) \rfloor - 1 + 2 + 1$ *Corollary 1*
$= \lfloor lg(n) \rfloor + 2$
□

(e) **Lemma 9:** $T(n) - T(1) = \sum_{k=1}^{n-1} D(k)$
**Direct Proof:**
$\sum_{k=1}^{n-1} D(k) = D(1) + D(2) + D(3) + ... + D(n-3) + D(n-2) + D(n-1)$
$= T(2) - T(1) + T(3) - T(2) + T(4) - T(3) +$
$... + T(n-2) - T(n-3) + T(n-1) - T(n-2) + T(n) - T(n-1)$
$= T(n) - T(1)$ *Due to cancellations*
□
**Corollary 2:**
By lemmas 9 and 8, $T(n) = \sum_{k=1}^{n-1} \lfloor (lg(n) + 2) \rfloor$

(f) **Lemma 10:** $T(n) \in O(nlog(n))$
$T(n) = \sum_{k=1}^{n-1} \lfloor (lg(n) + 2) \rfloor$
$= \sum_{k=1}^{n-1} \lfloor (lg(n)) \rfloor + \sum_{k=1}^{n-1} 2$
$\leq nlg(n) + 2n$
$\rightarrow T(n) \in O(nlog(n))$
□